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Abstract
We study the finite-temperature free energy and fermion number for Dirac fields in a one-dimensional spatial segment, under two different members of the family of local boundary conditions defining a self-adjoint Euclidean Dirac operator in two dimensions. For one of such boundary conditions, compatible with the presence of a spectral asymmetry, we discuss in detail the contribution of this part of the spectrum to the zeta-regularized determinant of the Dirac operator and, thus, to the finite-temperature properties of the theory.

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1. Introduction

When the Euclidean Dirac operator is considered on an even-dimensional compact manifold with boundary, its domain can be determined through a family of local boundary conditions which define a self-adjoint boundary problem [1] (the particular case of two-dimensional manifolds was first studied in [2]). The whole family is characterized by a real parameter θ, which can be interpreted as an analytic continuation of the well-known θ parameter in gauge theories. These boundary conditions can be considered to be the natural counterpart in Euclidean space of the well-known chiral bag boundary conditions.

Recently, it was shown [3] that the boundary problem so defined is not only self-adjoint, but also strongly elliptic [4, 5] in any even dimension. Also in [3], the meromorphic properties of the associated zeta function were determined for manifolds of the product type. For particular non-product manifolds, heat kernel coefficients and zeta functions were treated in [6]. Anomalies were also studied recently in [7].

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One salient characteristic of these local boundary conditions is the generation of an asymmetry in the spectrum of the Dirac operator. For the particular case of two-dimensional product manifolds, such asymmetry was shown, in [8], to be determined by the asymmetry of the boundary spectrum.

The aim of this paper is twofold:

(i) To discuss a physical application of Euclidean bag boundary conditions in two dimensions, with emphasis on the effect of the spectral asymmetry.

(ii) To (partially) answer the question posed in [9], as to whether the fermion number is modified by temperature in low-dimensional bags.

In section 2, we present the zero-temperature problem of Dirac fermions subject to local boundary conditions with arbitrary values of θ in $(1 + 1)$-dimensional Minkowski spacetime, and evaluate the vacuum energy and fermion number (equivalently, $U(1)$ charge). This model has recently been considered, in the context of brane theory, as the fermionic sector of an $N = 2$ supersymmetric sigma model, coupled to a magnetic field at the ends of the string [10].

In section 3, we determine the spectrum of the Euclidean Dirac operator at finite temperature for two particular values of θ. With these spectra at hand, we perform, in section 4, the calculation of the free energy via zeta function regularization.

Section 5 is devoted to the evaluation of the free energy in the grand-canonical ensemble and the fermion number, through the introduction of a chemical potential. (The finite-temperature fermion number for a different local boundary condition, not leading to a self-adjoint operator, was treated in [11].)

Finally, section 6 presents a discussion of the main results.

2. Definition of the problem in Minkowski spacetime

We will use the metric $(-, +)$, and choose for the Dirac matrices

$$\gamma_M^0 = i\sigma_1, \qquad \gamma_M^1 = -\sigma_2 \qquad \text{and} \qquad \tilde{\gamma}_M = \gamma_M^0 \gamma_M^1 = \sigma_3. \tag{2.1}$$

The action, for Dirac fermions coupled to a background field is given by

$$S_M = i \int d^2x \, \bar{\Psi}(i\slashed{\partial} - \slashed{A})\Psi. \tag{2.2}$$

Let us first particularize to the free case. The Hamiltonian can be determined from the classical equation of motion $i\slashed{\partial}\Psi = 0$, by proposing $\Psi(x^0, x^1) = e^{-iEx^0}\psi(x^1)$, with $0 \leqslant x^1 \leqslant L$. Thus, one gets the Hamiltonian $H = i\tilde{\gamma}_M\partial_1$. Its eigenfunctions are of the form

$$\psi(x^1) = \begin{pmatrix} A\,e^{-iEx^1} \\ B\,e^{iEx^1} \end{pmatrix}. \tag{2.3}$$

The boundary conditions will be taken to be

$$\tfrac{1}{2}\big(1 - i\gamma_M^0\,e^{i\tilde{\gamma}_M\theta_{0,L}}\big)\psi\big]_{x^1=0,L} = 0. \tag{2.4}$$

Now, it is easy to see that the eigenfunctions of the Hamiltonian depend only on the difference $\theta = \theta_L - \theta_0$, since the overall phase can always be eliminated through a constant chiral transformation. We will thus consider

$$\begin{aligned} \tfrac{1}{2}\big(1 - i\gamma_M^0\big)\psi\big]_{x^1=0} &= 0 \\ \tfrac{1}{2}\big(1 - i\gamma_M^0\,e^{i\tilde{\gamma}_M\theta}\big)\psi\big]_{x^1=L} &= 0. \end{aligned} \tag{2.5}$$

Once such boundary conditions are imposed, the eigenfunctions read

$$\psi_n(x^1) = \begin{pmatrix} A_n\, e^{-iE_n x^1} \\ -A_n\, e^{iE_n x^1} \end{pmatrix}, \tag{2.6}$$

where

$$E_n = \frac{n\pi}{L} + \frac{\theta}{2L} \qquad \text{with} \quad n = -\infty, \dots, \infty. \tag{2.7}$$

To evaluate the vacuum properties at zero temperature, let us first consider $0 < \theta < 2\pi$. Then, when defined through zeta function regularization [12], the Casimir energy is given by [13]

$$\begin{aligned}
E_C(\theta) &= -\frac{1}{2} \sum_n |E_n|^{-s} \Bigg|_{s=-1} \\
&= -\frac{\pi}{2L}\left(\zeta_H\left(-1, 1 - \frac{\theta}{2\pi}\right) + \zeta_H\left(-1, \frac{\theta}{2\pi}\right)\right) \\
&= \frac{\pi}{4L}\left(\frac{\theta^2}{2\pi^2} - \frac{\theta}{\pi} + \frac{1}{3}\right),
\end{aligned} \tag{2.8}$$

where $\zeta_H(s, u)$ is the Hurwitz zeta function [14].

In particular, $E_C(\pi) = -\frac{\pi}{24L}$. Note that, for $-2\pi < \theta < 0$, the replacement $\theta \to 2\pi + \theta$ must be performed. For $\theta = 0$

$$E_C(0) = -\frac{\pi}{L}\zeta_R(-1) = \frac{\pi}{12L}, \tag{2.9}$$

where ζ_R is the Riemann zeta function. So, the Casimir energy is continuous at $\theta = 0$.

As for the vacuum expectation value of the fermion number ($U(1)$ charge), again for $0 < \theta < 2\pi$, one has [15]

$$\begin{aligned}
N(\theta) &= -\frac{1}{2}\left(\sum_{E_n > 0} |E_n|^{-s} - \sum_{E_n < 0} |E_n|^{-s}\right)\Bigg|_{s=0} \\
&= \frac{1}{2}\left(\zeta_H\left(0, 1 - \frac{\theta}{2\pi}\right) - \zeta_H\left(0, \frac{\theta}{2\pi}\right)\right) \\
&= \frac{1}{2}\left(\frac{\theta}{\pi} - 1\right).
\end{aligned} \tag{2.10}$$

Also in this case, for $-2\pi < \theta < 0$ the replacement $\theta \to 2\pi + \theta$ must be performed. For $\theta = \pi$, one has $N(\pi) = 0$. But, at variance with the Casimir energy, the fermion number is discontinuous at $\theta = 0$. In fact, in this case, apart from a symmetric nonvanishing spectrum, a zero mode of the Hamiltonian appears, which is its own charge conjugate. As a consequence, $N(0) = \pm\frac{1}{2}$ [15].

In what follows, we will concentrate on two values of θ, i.e., $\theta = 0$ and $\theta = \pi$, to study the effect of a nonvanishing temperature on both vacuum quantities. In both cases, the Euclidean Dirac operator will turn to be self-adjoint, as shown in [1]. Note that these two boundary conditions are the ones corresponding to Ramond and Neveu–Schwarz strings [10].

3. Finite temperature: spectrum of the Dirac operator

In order to study the effect of temperature, we go to Euclidean space, with the metric (+, +). To this end, we take the Euclidean gamma matrices to be $\gamma_0 = -i\gamma_M^0 = \sigma_1$, $\gamma_1 = -\gamma_M^1 = \sigma_2$. Thus, the Euclidean action is

$$S_E = -iS_M = \int d^2x \, \bar{\Psi}(i\slashed{\partial} - \slashed{A})\Psi, \tag{3.1}$$

where the Euclidean zero component of the gauge potential (A_0) is related to the corresponding Minkowski one by $A_0 = -iA_0^M$.

We start by treating the free case; then, the partition function in the canonical ensemble is given by (see, for example, [20] and references therein)

$$\log Z = \log \det(i\slashed{\partial}_{BC}). \tag{3.2}$$

Here, BC stands for antiperiodic boundary conditions in the 'time' direction ($0 \leqslant x_0 \leqslant \beta$, with $\beta = \frac{1}{T}$) and, in the 'space' direction ($0 \leqslant x_1 \leqslant L$),

$$\tfrac{1}{2}(1 + \gamma_0)\Psi\rfloor_0 = 0 \qquad \tfrac{1}{2}(1 \pm \gamma_0)\Psi\rfloor_L = 0. \tag{3.3}$$

In the last equation, the plus sign corresponds to the case $\theta = 0$, while the minus sign corresponds to $\theta = \pi$.

In order to evaluate the partition function in the zeta regularization approach, we first determine the eigenfunctions, and the corresponding eigenvalues (ω), of the Dirac operator

$$i\slashed{\partial}\Psi = \begin{pmatrix} 0 & i\partial_0 + \partial_1 \\ i\partial_0 - \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \varphi(x_0, x_1) \\ \chi(x_0, x_1) \end{pmatrix} = \omega \begin{pmatrix} \varphi(x_0, x_1) \\ \chi(x_0, x_1) \end{pmatrix}. \tag{3.4}$$

To satisfy antiperiodic boundary conditions in the x_0 direction, we expand

$$\Psi(x_0, x_1) = \sum_\lambda e^{i\lambda x_0} \psi(x_1), \tag{3.5}$$

with

$$\lambda_l = (2l + 1)\frac{\pi}{\beta}, \qquad l = -\infty, \ldots, \infty. \tag{3.6}$$

After doing so we have, for each λ_l,

$$(-\lambda_l + \partial_1)\chi = \omega\varphi \qquad (-\lambda_l - \partial_1)\varphi = \omega\chi. \tag{3.7}$$

3.1. $\theta = 0$

It is easy to see that, with this boundary condition, no zero mode appears. For $\omega \neq 0$ one has, from (3.7),

$$\partial_1^2\varphi = -\kappa^2\varphi \qquad \chi = -\frac{1}{\omega}(\lambda_l + \partial_1)\varphi, \tag{3.8}$$

where $\kappa^2 = \omega^2 - \lambda_l^2$.

For $\kappa \neq 0$, for the eigenvalues one has

$$\omega_{n,l} = \pm\sqrt{\left(\frac{n\pi}{L}\right)^2 + \lambda_l^2}, \qquad \text{with} \quad n = 1, \ldots, \infty. \tag{3.9}$$

This part of the spectrum is symmetric. In the case $\kappa = 0$, one has a set of x_1-independent eigenfunctions, corresponding to

$$\omega_l = \lambda_l. \tag{3.10}$$

It is to be noted that $\omega_l = -\lambda_l$ are not eigenvalues; so, this last portion of the spectrum will be asymmetric or not, depending on whether the boundary spectrum ($\{\lambda_l\}$) is so. This was to be expected from our result in [8], where we proved that, in this case, the contribution of each boundary to the asymmetry equals one-half the boundary asymmetry, and the contributions from both boundaries add when the boundary conditions are the same at both of them. For λ_l as in (3.6), the asymmetry vanishes. However, the contribution of the spectral asymmetry will be shown to be crucial when evaluating the finite-temperature fermion number, which will be done in section 5.

3.2. $\theta = \pi$

Also, in this case, there are no zero modes.

At variance from the case $\theta = 0$, no solution exists for $\kappa = 0$, and only the symmetric part of the spectrum appears. The eigenvalues are given by

$$\omega_{n,l} = \pm\sqrt{\left(\left(n + \frac{1}{2}\right)\frac{\pi}{L}\right)^2 + \lambda_l^2}, \qquad \text{with} \quad n = 0, \ldots, \infty. \tag{3.11}$$

The absence of a nonsymmetric spectrum is also easy to understand from our result in [8], where it was shown that the sign of the boundary contribution to the asymmetry changes under a change of the intermediate sign in the projector defining the boundary conditions. So, in this case, the contributions from both boundaries cancel each other.

4. Free energy

With the eigenvalues of the Euclidean Dirac operator at hand, we can now obtain the partition function, which is

$$\log Z = \log \det(i\not{\partial}_{\text{BC}}). \tag{4.1}$$

When the determinant is defined through a zeta function regularization [12], one has

$$\log Z = -\frac{\text{d}}{\text{d}s}\bigg|_{s=0} \zeta\left(s, \frac{i\not{\partial}_{\text{BC}}}{\alpha}\right). \tag{4.2}$$

Here, $\zeta\left(s, \frac{i\not{\partial}_{\text{BC}}}{\alpha}\right)$ is the zeta function of the operator $\frac{i\not{\partial}_{\text{BC}}}{\alpha}$ [16] and, as usual, α is a parameter with dimensions of mass, introduced to render the zeta function dimensionless.

4.1. $\theta = 0$

We will first discuss in detail the case where the boundary conditions are determined by $\theta = 0$. We will then have two types of contributions. The first one comes from the symmetric part of the spectrum, equation (3.9); the second, from the 'nonsymmetric' part, equation (3.10). These two contributions are given by

$$\Delta_1 = -\frac{\text{d}}{\text{d}s}\bigg|_{s=0} \zeta_1(s), \tag{4.3}$$

where

$$\zeta_1(s) = (1 + (-1)^{-s}) \sum_{\substack{n=1 \\ l=-\infty}}^{\infty} \left[\left(\frac{n\pi}{\alpha L}\right)^2 + \left((2l+1)\frac{\pi}{\alpha\beta}\right)^2\right]^{-\frac{s}{2}}, \tag{4.4}$$

and

$$\Delta_2 = -\frac{d}{ds}\bigg|_{s=0} \zeta_2(s),\tag{4.5}$$

where

$$\zeta_2(s) = \sum_{l=-\infty}^{\infty} \left[(2l+1)\frac{\pi}{\alpha\beta}\right]^{-s}$$

$$= (1 + (-1)^{-s})\sum_{l=0}^{\infty}\left[(2l+1)\frac{\pi}{\alpha\beta}\right]^{-s}.\tag{4.6}$$

Note that, in equations (4.4) and (4.6), $(-1)^{-s}$ is undetermined (except at $s = 0$), since it can be taken to be $e^{\pm i\pi s}$, depending on the election of the cut when defining the complex power. As a consequence, the determinant could get an undetermined phase [17, 18]. We will come back to this point later on, when performing the s-derivatives of both zeta functions.

In order to do so, we must first perform an analytic extension of both expressions, which are convergent for $\Re(s)$ big enough. We first do this for ζ_1 (a general discussion of the procedure to be employed can be found, for instance, in [19]).

We write equation (4.4) as a Mellin transform

$$\zeta_1(s) = \frac{(1 + (-1)^{-s})}{\Gamma\left(\frac{s}{2}\right)}\int_0^\infty dt\, t^{\frac{s}{2}-1}\sum_{\substack{n=1 \\ l=-\infty}}^{\infty}\exp\left(-t\left[\left(\frac{n\pi}{\alpha L}\right)^2 + \left((2l+1)\frac{\pi}{\alpha\beta}\right)^2\right]\right).\tag{4.7}$$

Now, using the definition of the Jacobi theta function

$$\Theta_3(z, x) = \sum_{l=-\infty}^{\infty} e^{-\pi x l^2}\, e^{2\pi z l},\tag{4.8}$$

ζ_1 can be rewritten as

$$\zeta_1(s) = \frac{(1 + (-1)^{-s})}{(\sqrt{\pi})^s\Gamma\left(\frac{s}{2}\right)}\sum_{n=1}^{\infty}\int_0^\infty dt\, t^{\frac{s}{2}-1}\exp\left(-t\pi\left[\left(\frac{n}{\alpha L}\right)^2 + \left(\frac{1}{\alpha\beta}\right)^2\right]\right)$$

$$\times \Theta_3\left(\frac{-2t}{(\alpha\beta)^2}, \frac{4t}{(\alpha\beta)^2}\right).\tag{4.9}$$

To proceed, we will use the inversion formula for the Jacobi function

$$\Theta_3(z, x) = \frac{1}{\sqrt{x}}\exp\left(\frac{\pi z^2}{x}\right)\Theta_3\left(\frac{z}{ix}, \frac{1}{x}\right),\tag{4.10}$$

together with the definition (4.8), thus getting

$$\zeta_1(s) = \frac{(1 + (-1)^{-s})\beta\alpha}{2(\sqrt{\pi})^s\Gamma\left(\frac{s}{2}\right)}\sum_{n=1}^{\infty}\int_0^\infty dt\, t^{\frac{s-1}{2}-1}\exp\left(-t\pi\left(\frac{n}{\alpha L}\right)^2\right)$$

$$\times\left[1 + 2\sum_{l=1}^{\infty}\exp(i\pi l)\exp\left(-\frac{l^2\pi\beta^2\alpha^2}{4t}\right)\right].\tag{4.11}$$

Now, the integrals can be performed to get

$$\zeta_1(s) = \frac{(1 + (-1)^{-s})\beta}{2\alpha^{-s}(\sqrt{\pi})^s\Gamma\left(\frac{s}{2}\right)}\left[\Gamma\left(\frac{s-1}{2}\right)\frac{\pi^{\frac{1-s}{2}}}{L^{1-s}}\zeta_R(s-1)\right.$$

$$\left.+ 4\left(\frac{\beta L}{2}\right)^{\frac{s-1}{2}}\sum_{n,l=1}^{\infty}(-1)^l\left(\frac{l}{n}\right)^{\frac{s-1}{2}}K_{\frac{s-1}{2}}\left(\frac{nl\pi\beta}{L}\right)\right].\tag{4.12}$$

This completes the analytic extension of ζ_1. Note in particular that, due to the behaviour of the Bessel function $K_\nu(z)$ for large $|z|$ [14], the series in the second term between square brackets converges for all s.

We now extend ζ_2. This is quite a simple task. In fact, from the definition of the Hurwitz zeta function one can rewrite equation (4.6) as follows:

$$\zeta_2(s) = \left(\frac{2\pi}{\beta\alpha}\right)^{-s} (1 + (-1)^{-s})\zeta_H\left(s, \frac{1}{2}\right). \tag{4.13}$$

We can now go to the evaluation of both contributions (Δ_1 and Δ_2) to the logarithm of the determinant. Before actually doing so, note that ζ_1 and ζ_2 vanish for $s = 0$. This renders the ambiguity in defining $(-1)^{-s}$ irrelevant, even when taking the s-derivative. For the same reason, all dependence on the unphysical parameter α disappears from such derivative. So, it is easy to see, from equations (4.2), (4.3), (4.5), (4.12) and (4.13), that

$$\log Z = -\frac{\beta}{2}\left[\frac{\pi}{6L} + \frac{4}{\beta}\sum_{n,l=1}^{\infty}\frac{(-1)^l}{l}\exp\left(-\frac{nl\pi\beta}{L}\right)\right] + \log 2$$

$$= -\frac{\beta}{2}\left[\frac{\pi}{6L} - \frac{4}{\beta}\sum_{n=1}^{\infty}\log\left(1 + \exp\left(-\frac{n\pi\beta}{L}\right)\right)\right] + \log 2. \tag{4.14}$$

From this result, the free energy can be obtained as

$$F = -\frac{1}{\beta}\log Z = \frac{\pi}{12L} - \frac{2}{\beta}\sum_{n=1}^{\infty}\log\left(1 + \exp\left(-\frac{n\pi\beta}{L} =\right)\right) - \frac{1}{\beta}\log 2. \tag{4.15}$$

It is easy to see that, in the limit $\beta \to \infty$, the correct result for the Casimir energy at zero temperature (equation (2.9)) is obtained.

The high-temperature ($\beta \to 0$) behaviour of the free energy can be obtained by using the Euler–Maclaurin expansion, thus getting

$$F = -\frac{2L}{\pi\beta^2}\int_0^\infty dx \log(1 + e^{-x}) + O\left(\frac{\beta^2}{L^3}\right)$$

$$= -\frac{\pi L}{6\beta^2} + O\left(\frac{\beta^2}{L^3}\right). \tag{4.16}$$

Note that the contribution from ζ_2 cancels, in this limit, a term linear in the temperature appearing in the Euler–Maclaurin expansion.

4.2. $\theta = \pi$

In this case, there is only one contribution to the zeta function and, consequently, to the partition function, i.e.,

$$\zeta(s) = (1 + (-1)^{-s})\sum_{\substack{n=0 \\ l=-\infty}}^{\infty}\left[\left(\frac{(n + \frac{1}{2})\pi}{\alpha L}\right)^2 + \left((2l + 1)\frac{\pi}{\alpha\beta}\right)^2\right]^{-\frac{s}{2}}. \tag{4.17}$$

The steps leading to its analytic extension and to the evaluation of the partition function are essentially the same as in the previous subsection. As a result, one obtains

$$\zeta(s) = \frac{(1 + (-1)^{-s})\beta}{2\alpha^{-s}(\sqrt{\pi})^s \Gamma\left(\frac{s}{2}\right)} \left[\Gamma\left(\frac{s-1}{2}\right) \frac{\pi^{\frac{1-s}{2}}}{L^{1-s}} \zeta_H\left(s-1, \frac{1}{2}\right) \right.$$
$$\left. + 4\left(\frac{\beta L}{2}\right)^{\frac{s-1}{2}} \sum_{\substack{n=0 \\ l=1}}^{\infty} (-1)^l \left(\frac{l}{n+\frac{1}{2}}\right)^{\frac{s-1}{2}} K_{\frac{s-1}{2}}\left(\frac{(n+\frac{1}{2})l\pi\beta}{L}\right) \right], \tag{4.18}$$

and

$$\log Z = \frac{\pi\beta}{24L} + 2\sum_{n=0}^{\infty} \log\left(1 + \exp\left(-\frac{(n+\frac{1}{2})\pi\beta}{L}\right)\right). \tag{4.19}$$

Thus,

$$F = -\frac{\pi}{24L} - \frac{2}{\beta} \sum_{n=0}^{\infty} \log\left(1 + \exp\left(-\frac{(n+\frac{1}{2})\pi\beta}{L}\right)\right). \tag{4.20}$$

In the $\beta \to \infty$ limit, one re-obtains the Casimir energy in section 2. The high-temperature limit coincides with (4.16).

5. Fermion number

To evaluate the finite-temperature fermion number, we will introduce a small chemical potential (the meaning of 'small' will be specified, for each value of θ, in the corresponding subsection). Then, the finite-temperature fermion number will be calculated as

$$N = \frac{1}{\beta} \frac{\partial \log \mathcal{Z}}{\partial \mu}, \tag{5.1}$$

where $\mathcal{Z}$ is the grand-canonical partition function. In the language of thermostatistics, this is the mean particle number of the Fermi–Dirac gas in the grand-canonical ensemble. The answer to the question posed in [9] will be found from the value of this object at $\mu = 0$ which includes, for each value of θ, both the zero-temperature fermion number (given in equation (2.10) and the paragraph following it) and its temperature-dependent part.

We will follow [20] in introducing the chemical potential as an imaginary $A_0 = i\mu$ in Euclidean space (or, equivalently, a real A_0 in Minkowski spacetime). The Dirac eigenvalue equation then becomes

$$(i\slashed{\partial} + i\gamma_0\mu)\Psi = \omega\Psi, \tag{5.2}$$

while Ψ again satisfies antiperiodic boundary conditions in the 'time' direction, and the ones given by (3.3) in the 'space' direction.

The chemical potential can be eliminated from the differential equation through the transformation

$$\Psi = e^{-\mu x_0} \Psi'. \tag{5.3}$$

Thus, one gets for Ψ' the same differential equation as in the free case. Moreover, Ψ' satisfies the same 'spatial' boundary conditions but, in the 'time' direction, one has

$$\Psi'(\beta, x_1) = -e^{\mu\beta} \Psi'(0, x_1). \tag{5.4}$$

So, the effect of the chemical potential is to replace the 'temporal' eigenvalues in equation (3.6) with

$$\tilde{\lambda} = \lambda - i\mu = (2l + 1)\frac{\pi}{\beta} - i\mu, \qquad l = -\infty, \ldots, \infty. \tag{5.5}$$

5.1. $\theta = 0$

As before, we must consider two contributions to $\log \mathcal{Z}$

$$\Delta_1 = -\frac{d}{ds}\Big|_{s=0} \zeta_1(s), \tag{5.6}$$

where

$$\zeta_1(s) = (1 + (-1)^{-s}) \sum_{\substack{n=1 \\ l=-\infty}}^{\infty} \left[\left(\frac{n\pi}{\alpha L}\right)^2 + \left((2l + 1)\frac{\pi}{\alpha\beta} - i\frac{\mu}{\alpha}\right)^2 \right]^{-\frac{s}{2}}, \tag{5.7}$$

and

$$\Delta_2 = -\frac{d}{ds}\Big|_{s=0} \zeta_2(s), \tag{5.8}$$

where

$$\zeta_2(s) = \sum_{l=-\infty}^{\infty} \left[(2l + 1)\frac{\pi}{\alpha\beta} - i\frac{\mu}{\alpha} \right]^{-s}. \tag{5.9}$$

In order to perform the analytic extension of ζ_1, we will proceed much the same way as in the $\mu = 0$ case. However, in order to properly write (5.7) in terms of its Mellin transform, and freely interchange integrals and double series, we will take μ to satisfy $|\mu| < \frac{\pi}{L}$. Thus, we can write

$$\zeta_1(s) = \frac{(1 + (-1)^{-s})}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty dt\, t^{\frac{s}{2}-1} \sum_{\substack{n=1 \\ l=-\infty}}^{\infty} \exp\left(-t\left[\left(\frac{n\pi}{\alpha L}\right)^2 + \left((2l + 1)\frac{\pi}{\alpha\beta} - i\frac{\mu}{\alpha}\right)^2\right]\right). \tag{5.10}$$

This can also be written as

$$\zeta_1(s) = \frac{(1 + (-1)^{-s})}{(\sqrt{\pi})^s \Gamma\left(\frac{s}{2}\right)} \sum_{n=1}^\infty \int_0^\infty dt\, t^{\frac{s}{2}-1} \exp\left(-t\pi\left[\left(\frac{n}{\alpha L}\right)^2 + \left(\frac{1}{\alpha\beta} - \frac{i\mu}{\alpha\pi}\right)^2\right]\right)$$
$$\times \Theta_3\left(\frac{-2t}{\alpha\beta}\left(\frac{1}{\alpha\beta} - \frac{i\mu}{\alpha\pi}\right), \frac{4t}{(\alpha\beta)^2}\right). \tag{5.11}$$

From here on, the same steps as in the previous section can be followed to obtain

$$\zeta_1(s) = \frac{(1 + (-1)^{-s})\beta}{2\alpha^{-s}(\sqrt{\pi})^s \Gamma\left(\frac{s}{2}\right)} \left[\Gamma\left(\frac{s-1}{2}\right) \frac{\pi^{\frac{1-s}{2}}}{L^{1-s}} \zeta_R(s - 1) \right.$$
$$\left. + 4\left(\frac{\beta L}{2}\right)^{\frac{s-1}{2}} \sum_{n,l=1}^\infty (-1)^l \left(\frac{l}{n}\right)^{\frac{s-1}{2}} \cosh(\mu\beta l) K_{\frac{s-1}{2}}\left(\frac{nl\pi\beta}{L}\right) \right]. \tag{5.12}$$

Note that the double sum in the last term between square brackets is convergent in the range of μ considered. So, again, $\zeta_1(0) = 0$.

The analytic extension of ζ_2 requires, in this case, a more careful treatment than in the case $\mu = 0$. In fact,

$$
\begin{aligned}
\zeta_2(s) &= \sum_{l=-\infty}^{\infty} \left[(2l+1)\frac{\pi}{\alpha\beta} - i\frac{\mu}{\alpha} \right]^{-s} \\
&= \left(\frac{2\pi}{\alpha\beta} \right)^{-s} \left[\sum_{l=0}^{\infty} \left[\left(l + \frac{1}{2} \right) - i\frac{\mu\beta}{2\pi} \right]^{-s} + \sum_{l=0}^{\infty} \left[-\left(l + \frac{1}{2} \right) - i\frac{\mu\beta}{2\pi} \right]^{-s} \right] \\
&= \left(\frac{2\pi}{\alpha\beta} \right)^{-s} \left[\zeta_H \left(s, \frac{1}{2} - \frac{i\mu\beta}{2\pi} \right) + \sum_{l=0}^{\infty} \left[-\left(l + \frac{1}{2} \right) - i\frac{\mu\beta}{2\pi} \right]^{-s} \right].
\end{aligned}
\tag{5.13}
$$

Now, in order to write the second term as a Hurwitz zeta, we must relate the eigenvalues with negative real part to those with positive one without, in so doing, going through zeros in the argument of the power. Otherwise stated, we must select a cut in the complex ω plane [17]. This requirement determines a definite value of $(-1)^{-s}$, i.e., $(-1)^{-s} = \exp(i\pi \, \mathrm{sign}(\mu)s)$. Taking this into account, we finally have

$$
\zeta_2(s) = \left(\frac{2\pi}{\beta\alpha} \right)^{-s} \left[\zeta_H \left(s, \frac{1}{2} - \frac{i\mu\beta}{2\pi} \right) + \exp(i\pi \, \mathrm{sign}(\mu)s) \zeta_H \left(s, \frac{1}{2} + \frac{i\mu\beta}{2\pi} \right) \right].
\tag{5.14}
$$

From (5.12) and (5.14) both contributions to $\log \mathcal{Z}$ can be obtained. They are given by

$$
\Delta_1 = -\frac{\beta\pi}{12L} + \sum_{n=1}^{\infty} \log \left(1 + \exp \left(-\frac{2n\pi\beta}{L} \right) + 2\cosh(\mu\beta) \exp \left(-\frac{n\pi\beta}{L} \right) \right)
\tag{5.15}
$$

and

$$
\begin{aligned}
\Delta_2 &= -\left[\zeta_H' \left(0, \frac{1}{2} - \frac{i\mu\beta}{2\pi} \right) + \zeta_H' \left(0, \frac{1}{2} + \frac{i\mu\beta}{2\pi} \right) + i\pi \, \mathrm{sign}(\mu)\zeta_H \left(0, \frac{1}{2} + \frac{i\mu\beta}{2\pi} \right) \right] \\
&= \log 2 + \log \cosh \left(\frac{\mu\beta}{2} \right) - \frac{|\mu|\beta}{2}.
\end{aligned}
\tag{5.16}
$$

Both expressions can be seen to reduce to the corresponding ones in the previous section when $\mu = 0$.

Putting both pieces together, we finally have

$$
\begin{aligned}
\log \mathcal{Z} = &-\frac{\beta\pi}{12L} + \sum_{n=1}^{\infty} \log \left(1 + \exp \left(-\frac{2n\pi\beta}{L} \right) + 2\cosh(\mu\beta) \exp \left(-\frac{n\pi\beta}{L} \right) \right) \\
&+ \log 2 + \log \cosh \left(\frac{\mu\beta}{2} \right) - \frac{|\mu|\beta}{2}.
\end{aligned}
\tag{5.17}
$$

From this result, we can evaluate the free energy

$$
\begin{aligned}
F = &\frac{\pi}{12L} - \frac{1}{\beta} \left[\sum_{n=1}^{\infty} \log \left(1 + \exp \left(-\frac{2n\pi\beta}{L} \right) + 2\cosh(\mu\beta) \exp \left(-\frac{n\pi\beta}{L} \right) \right) \right. \\
&\left. + \log 2 + \log \cosh \left(\frac{\mu\beta}{2} \right) - \frac{|\mu|\beta}{2} \right].
\end{aligned}
\tag{5.18}
$$

It is easy to see that its $\beta \to \infty$ limit is $\frac{\pi}{12L}$, independently of the value of μ. This is consistent with the fact that the chemical potential has been introduced as a purely imaginary A_0 gauge potential in the Euclidean space. This corresponds to a real A_0 potential in the Minkowski spacetime. Now, as is well known, this last can be eliminated at zero temperature through a gauge transformation.

The high-temperature limit of the free energy is $-\frac{\pi L}{6\beta^2} - \frac{L\mu^2}{2\pi} + \frac{|\mu|}{2}$.

According to (5.1), the finite-temperature fermion number is given by

$$
N = \left\{ \sum_{n=1}^{\infty} \left[\frac{\exp\left(-\frac{n\pi\beta}{L} + \mu\beta\right)}{1 + \exp\left(-\frac{n\pi\beta}{L} + \mu\beta\right)} - \frac{\exp\left(-\frac{n\pi\beta}{L} - \mu\beta\right)}{1 + \exp\left(-\frac{n\pi\beta}{L} - \mu\beta\right)} \right] \right.
$$
$$
\left. + \frac{1}{2} \tanh\left(\frac{\mu\beta}{2}\right) - \frac{1}{2}\,\mathrm{sign}(\mu) \right\} .
\tag{5.19}
$$

In particular, for $\mu = 0$, it is clearly undefined. Note that this discontinuous character of the fermion number comes from zero 'spatial' eigenvalues or, equivalently, from the spectral asymmetry. More precisely, it originates from the phase of the determinant. We will comment on the interpretation of this result in the last section of the paper.

5.2. $\theta = \pi$

Since, for this value of θ, only the symmetric part of the spectrum contributes, we must consider the zeta function

$$
\zeta(s) = (1 + (-1)^{-s}) \sum_{\substack{n=0 \\ l=-\infty}}^{\infty} \left[\left(\frac{\left(n + \frac{1}{2}\right)\pi}{\alpha L}\right)^2 + \left((2l + 1)\frac{\pi}{\alpha\beta} - \mathrm{i}\frac{\mu}{\alpha}\right)^2 \right]^{-\frac{s}{2}},
\tag{5.20}
$$

whose analytic extension can be obtained with the same method as before, and is given (for $|\mu| < \frac{\pi}{2L}$) by

$$
\zeta(s) = \frac{(1 + (-1)^{-s})\beta}{2\alpha^{-s}(\sqrt{\pi})^s \Gamma\left(\frac{s}{2}\right)} \left[\Gamma\left(\frac{s-1}{2}\right) \frac{\pi^{\frac{1-s}{2}}}{L^{1-s}} \zeta_H\left(s - 1, \frac{1}{2}\right) \right.
$$
$$
\left. + 4\left(\frac{\beta L}{2}\right)^{\frac{s-1}{2}} \sum_{\substack{n=0 \\ l=1}}^{\infty} (-1)^l \left(\frac{l}{n + \frac{1}{2}}\right)^{\frac{s-1}{2}} \cosh(\mu\beta l) K_{\frac{s-1}{2}}\left(\frac{\left(n + \frac{1}{2}\right)l\pi\beta}{L}\right) \right].
\tag{5.21}
$$

The grand-canonical partition function, obtained by evaluating its s-derivative at $s = 0$ with reversed sign, gives us

$$
\log \mathcal{Z} = \frac{\beta\pi}{24L} + \sum_{n=0}^{\infty} \log\left(1 + \exp\left(-\frac{2\left(n + \frac{1}{2}\right)\pi\beta}{L}\right) + 2\cosh(\mu\beta)\exp\left(-\frac{\left(n + \frac{1}{2}\right)\pi\beta}{L}\right)\right).
\tag{5.22}
$$

In this case one has, for the free energy,

$$
F = -\frac{\pi}{24L} - \frac{1}{\beta} \sum_{n=0}^{\infty} \log\left(1 + \exp\left(-\frac{2\left(n + \frac{1}{2}\right)\pi\beta}{L}\right) + 2\cosh(\mu\beta)\exp\left(-\frac{\left(n + \frac{1}{2}\right)\pi\beta}{L}\right)\right).
\tag{5.23}
$$

Its low-temperature limit is $-\frac{\pi}{24L}$ which, again, is μ independent. The high-temperature limit is $-\frac{\pi L}{6\beta^2} - \frac{L\mu^2}{2\pi}$.

The fermion number is given by

$$N = \sum_{n=0}^{\infty} \left[\frac{\exp\left(-\frac{(n+\frac{1}{2})\pi\beta}{L} + \mu\beta\right)}{1 + \exp\left(-\frac{(n+\frac{1}{2})\pi\beta}{L} + \mu\beta\right)} - \frac{\exp\left(-\frac{(n+\frac{1}{2})\pi\beta}{L} - \mu\beta\right)}{1 + \exp\left(-\frac{(n+\frac{1}{2})\pi\beta}{L} - \mu\beta\right)} \right], \tag{5.24}$$

which vanishes for $\mu = 0$.

6. Discussion of the results

The result for the finite-temperature fermion number in the case $\theta = \pi$ (equation (5.24)) has a clear interpretation: no fermion number is created when rising the temperature, while keeping $\mu = 0$. Moreover, in this case, the low-temperature limit of the fermion number also vanishes for $\mu \neq 0$.

Now, in the case $\theta = 0$, the result in (5.19) requires a careful analysis. From the point of view of field theory, the right answer must be taken as one of the two possible limits. In fact, at zero temperature, a zero mode of the Hamiltonian appears in Minkowski space for $\mu = 0$ and, as already discussed (see section 2) there will be two nonequivalent vacuum states, with $N = \pm\frac{1}{2}$. Once the theory is quantized around one of these vacuum states, no extra fermion number ($U(1)$ charge) will arise at one loop, when compactifying the 'temporal' coordinate with an antiperiodic twist. This is, for $\theta = 0$, the answer to the question posed in [9]. However, the low-temperature limit of the fermion number can be seen to vanish if μ is kept different from zero, due to thermal averaging over the degenerate ground states.

It is interesting to note that our result for the partition function in this (Ramond) case does not coincide with the analytic extension of the one given, for instance, in chapter 10 of [21] (see also [10]), where the contribution from the phase of the determinant does not appear. Were one to disregard this term, the low-temperature limit of the fermion number would vanish for $\mu = 0$ (instead of picking one of the two possible vacuum values), while it would be $\frac{1}{2}\,\text{sign}(\mu)$ for $\mu \neq 0$.

Acknowledgments

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